Independent Study in

Dirac Propagator in External Field

Advisor: TING-WAI CHIU

By (Student): CHUN-CHUNG CHEN

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TABLE OF CONTENTS

I. Forewords

Propagators are very important in the calculation of physical systems. We can analytically or numerically evaluate the evolution of a physical system if we can evaluate its propagator. Therefore, we feel that we should calculate the propagator of a physical system before we can understand this system.

Generally, most of the propagators of physical systems are not already solved or can be easily solved. The purpose for this project is to do analytic calculations on Dirac propagator in external fields and see if we can get an exact solution or how far can we carry on the analysis.

II. Perturbation in Constant Magnetic Field

Dirac propagator in external EM field,

$$[i\not{\partial} - \mathrm{e}\not{A}(x) - m]S_F(x,x') = \delta^4(x-x'),$$

is usually solved by perturbation from free propagator,

$$(i\not{\partial} - m)S_F^0(x,x') = \delta^4(x-x').$$

$$\begin{aligned} S_F^0(x,x') &= \int d^4y S_F^0(x,y)[i\overrightarrow{\not{\partial}}_y - \mathrm{e}\not{A}(y) - m]S_F(y,x') \\ &= \int d^4y S_F^0(x,y)[-i\overleftarrow{\not{\partial}}_y - \mathrm{e}\not{A}(y) - m]S_F(y,x') \\ &= S_F(x,x') - \int d^4y S_F^0(x,y)\mathrm{e}\not{A}(y)S_F(y,x') \end{aligned}$$

(using $S_F^0(x,y)(-i\overleftarrow{\not{\partial}}_y - m) = \delta^4(x-y)$) We can derive the perturbation formula by iteration.

$$\begin{aligned} S_F(x,x') &= S_F^0(x,x') + \int d^4y S_F^0(x,y)\mathrm{e}\not{A}(y)S_F^0(y,x') \\ &+ \int\int d^4y_1 d^4y_2 S_F^0(x,y_1)\mathrm{e}\not{A}(y_1)S_F^0(y_1,y_2)\mathrm{e}\not{A}(y_2)S_F^0(y_2,x') + \cdots \end{aligned} \quad (2.1)$$

We try a simple case, massless propagator in uniform external magnetic field...

We know that the massless free propagator is

$$S_F^0 = \frac{\gamma(x-x')}{2\pi^2(x-x')^4}.$$

Assuming the vector potential to be

$$A^\mu = (0, Bx^{(2)}, 0, 0)$$

(Landau gauge for magnetic field in Z-direction) we can derive the first order correction for S_F from (2.1):

$$S_F(x,x') \approx \frac{\gamma(x-x')}{2\pi^2(x-x')^4} + \frac{\mathrm{e}}{4\pi^4}\gamma^\mu\gamma_\nu\gamma^\rho \int d^4y \frac{(x-y)_\mu A^\nu(y)(y-x')_\rho}{(x-y)^4(y-x')^4} \tag{2.2}$$

Let's calculate the integral that would appear when A is in first order of y:

$$\int d^4y \frac{(x-y)_\mu y_\nu (y-x')_\rho}{(x-y)^4(y-x')^4} = \int d^4y \frac{(l-y)_\mu (y+x')_\nu y_\rho}{(l-y)^4 y^4}$$

where $l \equiv x - x'$, $\quad y \to y + x'$. Use the Feynman formulas,

$$\begin{aligned}\frac{1}{ab} &= \int_0^1 d\theta \frac{1}{[a\theta + b(1-\theta)]^2} \\ \frac{1}{a^2b^2} &= \frac{\partial^2}{\partial a \partial b}\frac{1}{ab} = \int_0^1 d\theta \frac{6\theta(1-\theta)}{[a\theta+b(1-\theta)]^4},\end{aligned}$$

to combine the denominator.

$$\frac{1}{(l-y)^4y^4} = \int_0^1 d\theta \frac{6\theta(1-\theta)}{[(l-y)^2\theta + y^2(1-\theta)]^4} = \int_0^1 d\theta \frac{6\theta(1-\theta)}{[(y-\theta l)^2 + l^2\theta(1-\theta)]^4}$$

Make a change of variable: $y \to y + \theta l$. The integral becomes

$$\int_0^1 d\theta\, 6\theta(1-\theta) \int d^4y \frac{(l-y)_\mu (y+x')_\nu y_\rho}{[y^2 + l^2\theta(1-\theta)]^4}.$$

Expand the numerator and integrate separately. Because of the null results from the integrations of odd functions, the only terms remain are

$$\begin{aligned} Q &\equiv \int d^4y \frac{1}{[y^2 + l^2\theta(1-\theta)]^4} \\ P_{\mu\nu} &\equiv \int d^4y \frac{y_\mu y_\nu}{[y^2+l^2\theta(1-\theta)]^4} \\ &= g_{\mu\nu}P. \end{aligned}$$

The integration becomes

$$\int_0^1 d\theta\, 6\theta(1-\theta)[(1-\theta)l_\mu P_{\nu\rho} - (x'+\theta l)_\nu P_{\mu\rho} - \theta l_\rho P_{\mu\nu} + (1-\theta)l_\mu (x'+\theta l)_\nu \theta l_\rho Q]$$

To integrate P, Q , we must rotate the contour of y_0 integral. We should notice that the poles selection in Feynman propagator is equivalent to the rotation of integration contour of p_0 into $-i\infty \to i\infty$. Therefore, to keep $p \cdot x$ invariant, we must rotate y_0 integral into $i\infty \to -i\infty$. That is, replace y_0 by $y_4 = iy_0$

$$\begin{aligned} \int d^4y &\to -i\int d^4y \\ y^2 = y_0^2 - y_1^2 - y_2^2 - y_3^2 &\to -y^2 = -y_1^2 - y_2^2 - y_3^2 - y_4^2 \end{aligned}$$

By letting $c = l^2\theta(1-\theta)$, we get

$$\begin{aligned}
P_\mu{}^\mu = 4P &= i\int d^4y \frac{y^2}{(y^2-c)^4} \\
&= i\int r^3 dr \sin^2\theta_1 d\theta_1 \sin\theta_2 d\theta_2 d\theta_3 \frac{r^2}{(r^2-c)^4} \\
&= i2\pi^2 \int_0^\infty dr \frac{r^5}{(r^2-c)^4} = i\pi^2 \int_0^\infty dr^2 \frac{r^4}{(r^2-c)^4} \\
&= i\pi^2 \int_{-c}^\infty dr \frac{(r+c)^2}{(r-c)^4}, \qquad r^2 - c \to r \\
&= \frac{-i\pi^2}{3c} \\
P &= \frac{-i\pi^2}{12c}
\end{aligned}$$

In a similar way, we find

$$Q = \frac{-i\pi^2}{6c^2}$$

and

$$\begin{aligned}
&\int d^4y \frac{(x-y)_\mu y_\nu (y-x')_\rho}{(x-y)^4(y-x')^4} \\
&= \int_0^1 d\theta\, 6\theta(1-\theta)\Big\{[(1-\theta)l_\mu g_{\nu\rho} - (x'+\theta l)_\nu g_{\mu\rho} - \theta l_\rho g_{\mu\nu}]\frac{-i\pi^2}{12l^2\theta(1-\theta)} \\
&\quad -(1-\theta)l_\mu(x'+\theta l)_\nu \theta l_\rho \frac{i\pi^2}{6l^4\theta^2(1-\theta)^2}\Big\} \\
&= \frac{-i\pi}{4(x-x')^4}\Big\{[(x-x')_\mu g_{\nu\rho} - (x+x')_\nu g_{\mu\rho} - (x-x')_\rho g_{\mu\nu}](x-x')^2 \\
&\quad +2(x-x')_\mu(x+x')_\nu(x-x')_\rho\Big\}.
\end{aligned}$$

Substitute the actual A and the above result into (2.2)

$$\begin{aligned}
S_F(x,x') &\approx S_F^0 + \frac{\mathrm{e}B}{4\pi^4}\gamma^\mu\gamma_1\gamma^\rho \int d^4y \frac{(x-y)_\mu y^{(2)}(y-x')_\rho}{(x-y)^4(y-x')^4} \\
&= S_F^0 - \frac{\mathrm{e}Bi}{(4\pi)^2(x-x')^4}\gamma^\mu\gamma_1\gamma^\rho\{[(x-x')_\mu g_\rho^2 - (x+x')_\nu g_{\mu\rho} \\
&\quad -(x-x')_\rho g_\mu^2](x-x')^2 + 2(x-x')_\mu(x+x')^{(2)}(x-x')_\rho\} \\
&= S_F^0 - \frac{\mathrm{e}Bi}{(4\pi)^2(x-x')^4}\{(x-x')^2[(\gamma^\mu\gamma_1\gamma^2 - \gamma^2\gamma_1\gamma^\mu)(x-x')_\mu + 2\gamma_1(x+x')_{(2)}] \\
&\quad +4(x-x')_1(x+x')^{(2)}\gamma^\mu(x-x')_\mu - 2\gamma_1(x+x')^{(2)}(x-x')^2\}.
\end{aligned}$$

Use the anticommutators of γ,

$$\gamma^2\gamma_1\gamma^\mu = 2\gamma^2 g_1^\mu - \gamma^2\gamma^\mu\gamma_1 = 2\gamma^2 g_1^\mu - g^{2\mu}\gamma_1 - \gamma^\mu\gamma^2\gamma_1.$$

We move $\gamma(x-x')$ ahead and get

$$\begin{aligned} S_F(x,x') \;\approx\; & \frac{\gamma(x-x')}{2\pi^2(x-x')^4} - \frac{i\mathrm{e}B\gamma(x-x')(x-x')_1(x+x')^{(2)}}{4\pi^2(x-x')^4} + \\ & \frac{i\mathrm{e}B[\gamma^2(x-x')_1 - \gamma_1(x-x')^{(2)}]}{8\pi^2(x-x')^2} + \frac{i\mathrm{e}B\gamma(x-x')[\gamma^2,\gamma_1]}{16\pi^2(x-x')^2} \end{aligned}$$

after substitutions and reductions. This is the first order propagator of massless Dirac particle in constant magnetic field!

III. Proper Time Method

Proper time method is used to solve the linear differential equations for Green's functions, that is, the propagators of physical systems. The Green's function, $G(x,x')$, of a linear differential operator, $H(x,i\partial)$, satisfies

$$H(x,i\partial)G(x,x') = \delta(x-x').$$

We can regard $G(x,x')$ as the matrix element of an operator

$$G(x,x') = \langle x|G|x'\rangle$$

and introduce position operator, $\langle x|x|x'\rangle = x\delta(x-x')$, and momentum operator,$\langle x|p|x'\rangle = i\partial\delta(x-x') = -i\partial'\delta(x-x')$. (Space-time coordinates become a hermitian operator with eigenstate $|x\rangle$ satisfying $\langle x|x'\rangle = \delta(x-x')$).

The original equation becomes the x-representation of the operator equation,

$$H(x,p)G = 1.$$

Our job becomes to solve for the matrix elements of the inverse operator of $H(x,p)$.

For we not knowing the matrix element of $H(x,p)^{-1}$, we are trying to represent it by other operators that we have known their matrix element. For instance,

$$\begin{aligned} G &= (\not{p} - \mathrm{e}\not{A} - m)^{-1} = (\not{p} - m)^{-1}[1 - \mathrm{e}\not{A}\,(\not{p} - m)^{-1}]^{-1} \\ &= (\not{p} - m)^{-1}\sum_{n=0}^{\infty}[\mathrm{e}\not{A}\,(\not{p} - m)^{-1}]^n \\ &= (\not{p} - m)^{-1} + (\not{p} - m)^{-1}\mathrm{e}\not{A}\,(\not{p} - m)^{-1} + (\not{p} - m)^{-1}\mathrm{e}\not{A}\,(\not{p} - m)^{-1}\mathrm{e}\not{A}\,(\not{p} - m)^{-1} + \cdots \end{aligned}$$

where the matrix element of $(\not{p} - m)^{-1}$ is just the free propagator of electron and the matrix multiplication is just the volume integral of space-time coordinates. This results in the well known perturbation method!

The proper time method is using the transformation formula,

$$i\int_0^\infty e^{-i\mathrm{A}s}\,ds = i(-i\mathrm{A})^{-1}e^{-i\mathrm{A}s}\Big|_{s=0}^{s=\infty} = \mathrm{A}^{-1}, \tag{3.1}$$

to represent $H(x,p)^{-1}$ with $\exp[-iH(x,p)s]$, solving the matrix element and integrating with respect to s to furnish $G(x,x')$. We call $\exp[-iH(x,p)s]$ the evolution operator, $U(s)$, of the "Proper Time", s. The definitions of the proper time evolutions for general operators and states are:

$$\begin{aligned} U(s) &\equiv e^{-iH(x,p)s} \\ \mathcal{O}(s) &\equiv U^{-1}(s)\mathcal{O}U(s) \\ |\varphi(s)\rangle &\equiv U^{-1}(s)|\varphi\rangle \end{aligned}$$

We can proof the following formulas:

$$\begin{gathered} H[x(s),p(s)] = U^{-1}(s)H(x,p)U(s) = H(x,p) \\ i\frac{\partial}{\partial s}U(s) = H(x,p)U(s) \\ \frac{\partial}{\partial s}\mathcal{O}(s) = iH(x,p)U^{-1}(s)\mathcal{O}U(s) - U^{-1}(s)\mathcal{O}iH(x,p)U(s) = i[H(x,p),\mathcal{O}(s)] \\ [p(s),x(s)] = U^{-1}(s)[p,x]U(s) = ig \\ \lim_{s\to 0}\langle x|U(s)|x'\rangle = \langle x|x'\rangle = \delta(x-x') \\ \langle x|U(s)|x'\rangle = \langle x(s)|x'(0)\rangle \end{gathered} \tag{3.2}$$

If we can use these formulas to represent $p(s)$ with $x(s)$ and $x(0)$ and arrange $H[x(s),p(s)]$ into a polynomial, $F[x(s),x(0);s]$, where $x(s)$ goes before $x(0)$, we can get the equation that the matrix element of the evolution operator must satisfy:

$$\begin{aligned} i\frac{\partial}{\partial s}\langle x(s)|x'(0)\rangle &= \langle x(s)|H[x(s),p(s)]|x'(0)\rangle = \langle x(s)|F[x(s),x(0);s]|x'(0)\rangle \\ &= F(x,x';s)\langle x(s)|x'(0)\rangle \end{aligned} \tag{3.3}$$

Integrate it into

$$\langle x(s)|x'(0)\rangle = \exp\left[-i\int^{s} ds' F(x,x';s')\right]C(x,x'), \tag{3.4}$$

use the foregoing conditions to fix $C(x,x')$ and what remains is the integration with respect to the proper time 's':

$$\begin{aligned} G(x,x') &= \langle x|H(x,p)^{-1}|x'\rangle = i\int_0^\infty ds\langle x|e^{-H(x,p)s}|x'\rangle \\ &= i\int_0^\infty ds\langle x(s)|x'(0)\rangle \end{aligned}$$

IV. Dirac Propagator in External EM Field

The Dirac field equations and commutator is

$$\begin{gathered} (i\partial\!\!\!/ - \mathrm{e}A\!\!\!/ - m)\psi = 0, \\ \bar{\psi}(-i\overleftarrow{\partial\!\!\!/} - \mathrm{e}A\!\!\!/ - m) = 0, \\ \{\psi(\mathrm{x},t),\bar{\psi}(\mathrm{x}',t)\} = \gamma_0\delta(\mathrm{x}-\mathrm{x}'), \end{gathered} \tag{4.1}$$

and the Feynman propagator of Dirac field is

$$\begin{aligned} S_F(x,x') &= i\langle 0|T(\psi(x)\bar{\psi}(x'))|0\rangle \\ &= i\theta(x_0-x'_0)\langle 0|\psi(x)\bar{\psi}(x')|0\rangle - i\theta(x'_0-x_0)\langle 0|\bar{\psi}(x')\psi(x)|0\rangle \end{aligned}$$

$$\theta(x) \equiv \begin{cases} 1, & x>0 \\ 0, & x\leq 0 \end{cases}$$

According to (4.1), we know it's a Green's function of Dirac equation:

$$\begin{aligned} (i\not\partial - \mathrm{e}\not A - m)S_F(x,x') &= \gamma_0\langle 0|\{\psi(x),\bar{\psi}(x')\}|0\rangle\delta(x_0-x'_0), \\ &= \delta(x-x'); \end{aligned}$$

In the case of free field, we can get it by differential of scaler propagator:

$$\begin{aligned} S_F(x,x') &= -(i\not\partial + m)G(x,x') \\ &= -(i\not\partial + m)\frac{-1}{(2\pi)^4}\int d^4p\frac{1}{p^2-m^2+i\varepsilon}e^{-ip\cdot(x-x')} \end{aligned} \tag{4.2}$$

The purpose of the ε here is to select the correct pole (The pole is $p=-m$ for $t>t'$ and $p=m$ for $t<t'$) to meet the requirement of time ordering.

Of cause, we can let $U(s)=\exp[-i\not\pi]$ to calculate the propagator $G=(\not\pi-m)^{-1}$ in external EM field. (where $\pi=p-\mathrm{e}A$) But, in this manner, we are unable to represent $p(s)$ (or $\pi(s)$) in terms of $x(s)$ and $x(0)$! Just as the free propagator , we come to the matrix element of $G=[(m-\not\pi)(m+\not\pi)]^{-1}=(m^2-\not\pi^2)^{-1}$ (just like the scaler propagator above) than apply $-(\not\pi+m)$ on $G(x,x')$ to get the desired result.

We define the H and reduce the γ matrix.

$$\begin{aligned} \not\pi^2 &= \gamma^\mu\gamma^\nu\pi_\mu\pi_\nu = \gamma^\nu\gamma^\mu(\pi_\mu\pi_\nu-[\pi_\mu,\pi_\nu]) = \frac{1}{2}\{\gamma^\mu,\gamma^\nu\}\pi_\mu\pi_\nu - \frac{1}{2}\gamma^\nu\gamma^\mu[\pi_\mu,\pi_\nu] \\ &= \pi^2 - \frac{-1}{4}[\gamma^\mu,\gamma^\nu][\pi_\mu,\pi_\nu] \\ &= \pi^2 - \frac{\mathrm{e}}{2}\sigma_{\mu\nu}F^{\mu\nu} \end{aligned}$$

$$H \equiv m^2 - \not\pi^2 = m^2 + \frac{\mathrm{e}}{2}\sigma_{\mu\nu}F^{\mu\nu} - \pi^2$$

using $\sigma^{\mu\nu}\equiv\frac{i}{2}[\gamma^\mu,\gamma^\nu]$ and $[\pi_\mu,\pi_\nu]=[i\partial_\mu-\mathrm{e}A_\mu, i\partial_\nu-\mathrm{e}A_\nu]=-i\mathrm{e}F_{\mu\nu}$. (We define H as $m^2-\not\pi^2$ instead of $\not\pi^2-m^2$ because of the unwritten $i\varepsilon$ following the m^2(refer to (4.2)). In such arrangement, e^{-iHs} will approach zero under the transformation of (3.1) as $s\to\infty$!) By the formula (3.2), we can get the equation of motion.

$$\begin{aligned} \frac{d}{ds}x_\mu(s) &= i[H,x_\mu(s)] = -i[\pi^2(s),x_\mu(s)] \\ &= -i\pi^\nu(s)[\pi_\nu(s),x_\mu(s)] - i[\pi^\nu(s),x_\mu(s)]\pi_\nu(s) = -i\pi^\nu(s)ig_{\nu\mu} - iig^\nu_\mu\pi_\nu(s) \\ &= 2\pi_\mu(s) \end{aligned} \tag{4.3}$$

$$
\begin{aligned}
\frac{d}{ds}\pi_\mu(s) &= i[H,\pi_\mu(s)] = i[-\pi^2(s) + \frac{\mathrm{e}}{2}\sigma_{\nu\rho}F^{\nu\rho}[x(s)],\pi_\mu(s)] \\
&= -i\pi^\nu(s)[\pi_\nu(s),\pi_\mu(s)] - i[\pi_\nu(s),\pi_\mu(s)]\pi^\nu(s) + \frac{i\mathrm{e}}{2}\sigma_{\nu\rho}[F^{\nu\rho}[x(s)],\pi_\mu(s)] \\
&= \mathrm{e}\pi^\nu(s)F_{\mu\nu}[x(s)] + \mathrm{e}F_{\mu\nu}[x(s)]\pi^\nu(s) + \frac{\mathrm{e}}{2}\sigma^{\nu\rho}\partial_\mu F_{\nu\rho}[x(s)] \\
&= 2\mathrm{e}F_{\mu\nu}[x(s)]\pi^\nu(s) - [\mathrm{e}F_{\mu\nu}[x(s)],\pi^\nu(s)] + \frac{\mathrm{e}}{2}\sigma^{\nu\rho}\partial_\mu F_{\nu\rho}[x(s)] \\
&= 2\mathrm{e}F_{\mu\nu}[x(s)]\pi^\nu(s) + i\mathrm{e}\partial_\nu F_{\mu\nu}[x(s)] + \frac{\mathrm{e}}{2}\sigma^{\nu\rho}\partial_\mu F_{\nu\rho}[x(s)] \qquad (4.4)
\end{aligned}
$$

By using these equations, we can represent H in terms of $x(s)$, $x(0)$ in some cases and construct more convenient conditions for $C(x,x')$ from (3.2).

$$
\begin{aligned}
[i\partial^x - \mathrm{e}A(x)]\langle x(s)|x'(0)\rangle &= \langle x(s)|\pi(s)|x'(0)\rangle \\
[-i\partial^{x'} - \mathrm{e}A(x')]\langle x(s)|x'(0)\rangle &= \langle x(s)|\pi(0)|x'(0)\rangle. \qquad (4.5)
\end{aligned}
$$

The Dirac propagator becomes

$$
\begin{aligned}
S_F(x,x') &= -[i\not\partial - \mathrm{e}\not A(x) + m]i\int_0^\infty ds\langle x(s)|x'(0)\rangle \\
&\quad -i\int_0^\infty ds[\langle x(s)|\pi(s)|x'(0)\rangle + m\langle x(s)|x'(0)\rangle] \qquad (4.6)
\end{aligned}
$$

V. Constant Field

In the case of constant field, (4.3) and (4.4) reduce to

$$
\frac{d}{ds}x(s) = 2\pi(s), \quad \frac{d}{ds}\pi(s) = 2\mathrm{e}F\pi(s)
$$

We skiped the summation index between tensors of rank two or rank one.(That is, regard them as four dimensional matrixes and column or row vectors. See appendix.) We can get

$$
\begin{aligned}
\pi(s) &= e^{2\mathrm{e}Fs}\pi(0) \\
x(s) - x(0) &= \frac{e^{2\mathrm{e}Fs}-1}{\mathrm{e}F}\pi(0)
\end{aligned} \qquad (5.1)
$$

from the equations of motion. Therefore,

$$
\pi(s) = \frac{\mathrm{e}F\, e^{\mathrm{e}Fs}}{2\sinh(\mathrm{e}Fs)}[x(s)-x(0)]. \qquad (5.2)
$$

Because F is antisymmetry, we have

$$
\begin{aligned}
\pi^2(s) &= \frac{1}{4}[x(s)-x(0)]\frac{-\mathrm{e}F\, e^{-\mathrm{e}Fs}}{2\sinh(-\mathrm{e}Fs)}\frac{\mathrm{e}F\, e^{\mathrm{e}Fs}}{2\sinh(\mathrm{e}Fs)}[x(s)-x(0)] \\
&= [x(s)-x(0)]K[x(s)-x(0)]
\end{aligned}
$$

where $K \equiv \frac{1}{4}(\mathrm{e}F)^2 \sinh^{-2}(\mathrm{e}Fs)$. Use the commutator,

$$\begin{aligned}
[x_\mu(s), x_\nu(0)] &= \Big[\Big(\frac{e^{2\mathrm{e}Fs}-1}{\mathrm{e}F}\Big)_\mu^{\ \rho} \pi_\rho(0) + x_\mu(0), x_\nu(0)\Big] \\
&= \Big(\frac{e^{2\mathrm{e}Fs}-1}{\mathrm{e}F}\Big)_\mu^{\ \rho} [\pi_\rho(0), x_\nu(0)] = \Big(\frac{e^{2\mathrm{e}Fs}-1}{\mathrm{e}F}\Big)_\mu^{\ \rho} i g_{\rho\mu} \\
&= i\Big(\frac{e^{2\mathrm{e}Fs}-1}{\mathrm{e}F}\Big)_{\mu\nu},
\end{aligned}$$

to move $x(s)$ ahead of $x(0)$.

$$\begin{aligned}
K^{\mu\nu}[x_\nu(s), x_\mu(0)] &= i\mathrm{tr}\Big(\frac{(\mathrm{e}F)^2}{4\sinh^2(\mathrm{e}Fs)} \frac{e^{2\mathrm{e}Fs}-1}{\mathrm{e}F}\Big) = i\mathrm{tr}\Big(\frac{\mathrm{e}F}{2}\frac{e^{\mathrm{e}Fs}+e^{-\mathrm{e}Fs}}{e^{\mathrm{e}Fs}-e^{-\mathrm{e}Fs}} + \frac{\mathrm{e}F}{2}\Big) \\
&= \frac{i}{2}\mathrm{tr}[\mathrm{e}F\coth(\mathrm{e}Fs)]
\end{aligned}$$

$$\begin{aligned}
\pi^2(s) &= K^{\mu\nu}[x_\mu(s)x_\nu(s) + x_\mu(0)x_\nu(0) - x_\mu(s)x_\nu(0) - x_\mu(0)x_\nu(s)] \\
&= K^{\mu\nu}[x_\mu(s)x_\nu(s) + x_\mu(0)x_\nu(0) - 2x_\mu(s)x_\nu(0)] + \frac{i}{2}\mathrm{tr}[\mathrm{e}F\coth(\mathrm{e}Fs)]
\end{aligned}$$

We used the conditions that K is symmetric and F is traceless. Use them to represent H.

$$H = m^2 + \frac{e}{2}(\sigma\cdot F) - x(s)Kx(0) + 2x(s)Kx(0) - x(0)Kx(0) - \frac{i}{2}\mathrm{tr}[\mathrm{e}F\coth(\mathrm{e}Fs)]$$

This is just the $F(x(s), x(0); s)$ in (3.3). We can derive

$$\begin{aligned}
\langle x(s)|x'(0)\rangle &= C(x,x')\exp\big[-i\int^s ds' F(x,x';s')\big] \\
&= C(x,x')\exp\Big(-i\int^s ds\Big\{m^2 + \frac{\mathrm{e}}{2}(\sigma\cdot F) - (x-x')K(x-x') \\
&\quad -\frac{i}{2}\mathrm{tr}[\mathrm{e}F\coth(\mathrm{e}Fs)]\Big\}\Big) \\
&= C(x,x')\exp\Big\{-im^2s - i\frac{\mathrm{e}}{2}(\sigma\cdot F)s + i(x-x')\int^s ds\frac{(\mathrm{e}F)^2}{4\sinh^2(\mathrm{e}Fs)}(x-x') \\
&\quad -\frac{1}{2}\mathrm{tr}\int^s ds[\mathrm{e}F\coth(\mathrm{e}Fs)]\Big\} \\
&= C(x,x')\exp\Big\{-im^2s - i\frac{\mathrm{e}}{2}(\sigma\cdot F)s - \frac{i}{4}(x-x')\mathrm{e}F\coth(\mathrm{e}Fs)(x-x') \\
&\quad -\frac{1}{2}\mathrm{tr}[\ln\frac{\sinh(\mathrm{e}Fs)}{\mathrm{e}Fs} + \ln s]\Big\} \\
&= \frac{1}{s^2}C(x,x')\exp\Big\{-\frac{1}{2}\mathrm{tr}\ln\frac{\sinh(\mathrm{e}Fs)}{\mathrm{e}Fs} - \frac{i}{4}(x-x')\mathrm{e}F\coth(\mathrm{e}Fs)(x-x') \\
&\quad -im^2s - i\frac{\mathrm{e}}{2}(\sigma\cdot F)s\Big\}
\end{aligned}$$

from (3.4). Substituting (5.1) and (5.2) into the first equation of (4.5), we get the equation which $C(x,x')$ satisfies:

$$\begin{aligned}
&[i\partial^x - \mathrm{e}A(x)]C(x,x')\exp\big[-\frac{i}{4}(x-x')\mathrm{e}F\coth(\mathrm{e}Fs)(x-x')+\cdots\big] \\
&= -\frac{\mathrm{e}F\,e^{\mathrm{e}Fs}}{2\sinh(\mathrm{e}Fs)}(x-x')C(x,x')\exp\big[-\frac{i}{4}(x-x')\mathrm{e}F\coth(\mathrm{e}Fs)(x-x')+\cdots\big] \\
\Rightarrow\ &[i\partial^x - \mathrm{e}A(x) + \frac{\mathrm{e}F\,e^{\mathrm{e}Fs}}{2\sinh(\mathrm{e}Fs)}]C(x,x')\exp\big[-\frac{i}{4}(x-x')\mathrm{e}F\coth(\mathrm{e}Fs)(x-x')\big] = 0 \\
\Rightarrow\ &[i\partial^x - \mathrm{e}A(x) + \frac{\mathrm{e}F\,e^{\mathrm{e}Fs}}{2\sinh(\mathrm{e}Fs)} + \frac{1}{2}\mathrm{e}F\coth(\mathrm{e}Fs)(x-x')]C(x,x') = 0 \\
\Rightarrow\ &[i\partial^x - \mathrm{e}A(x) - \frac{\mathrm{e}F}{2}(x-x')]C(x,x') = 0
\end{aligned}$$

(mist. "$\cdots$" is the part independent of x). Similarly we can substitute them into the second equation of (4.5):

$$[-i\partial^{x'} - \mathrm{e}A(x') + \frac{\mathrm{e}F}{2}(x-x')]C(x,x') = 0$$

Use them to fix $C(x,x')$:

$$C(x,x') = C\exp\Big\{-i\mathrm{e}\int_{x'}^{x} d\xi[A(\xi) + \frac{1}{2}F(\xi - x')]\Big\}$$

C can be fixed by using the fifth equation in (3.2). As $s \to 0$,

$$\begin{aligned}
\langle x(s)|x'(0)\rangle &= \frac{C}{s^2}\exp\Big\{-ie\int_{x'}^{x} d\xi[A(\xi)+\frac{1}{2}F(\xi-x')] - \frac{1}{2}\mathrm{tr}\ln\frac{\sinh(\mathrm{e}Fs)}{\mathrm{e}Fs} \\
&\quad -\frac{i}{4}(x-x')\mathrm{e}F\coth(\mathrm{e}Fs)(x-x') - im^2 s - i\frac{e}{2}(\sigma\cdot F)s\Big\} \\
&\to \frac{C}{s^2}\exp\big[-\frac{i(x-x')^2}{4s}\big].
\end{aligned}$$

Because this should become δ-function, we have

$$\begin{aligned}
1 &= \int d^4x\frac{C}{s^2}\exp\big[-\frac{i(x-x')^2}{4s}\big] \\
&= \frac{C}{s^2}\cdot\frac{1-i}{\sqrt{2}}\sqrt{4\pi s}\cdot(\frac{1+i}{\sqrt{2}}\sqrt{4\pi s})^3 \\
&= i(4\pi)^2 C
\end{aligned}$$

and $C = -i(4\pi)^{-2}$. So,

$$\begin{aligned}
\langle x(s)|x'(0)\rangle = \frac{-i}{(4\pi)^2 s^2}\exp\Big\{&-i\mathrm{e}\int_{x'}^{x} d\xi[A(\xi)+\frac{1}{2}F(\xi-x')] - \frac{1}{2}\mathrm{tr}\ln\frac{\sinh(\mathrm{e}Fs)}{\mathrm{e}Fs} \\
&-\frac{i}{4}(x-x')\mathrm{e}F\coth(\mathrm{e}Fs)(x-x') - im^2 s - \frac{i\mathrm{e}}{2}(\sigma\cdot F)s\Big\}.
\end{aligned} \tag{5.3}$$

If we restrict the integration along straight line, we have $\int d\xi[A(\xi)+\frac{1}{2}F(\xi-x')]=\int d\xi A(\xi)$ and

$$\begin{aligned} S_F &= -(i\partial\!\!\!/ - \mathrm{e}A\!\!\!/ + m)i\int_0^\infty ds\langle x(s)|x'(s)\rangle &(5.4)\\ &= -i\int_0^\infty ds\langle x(s)|(\pi\!\!\!/ + m)|x'(s)\rangle \\ &= -i\int_0^\infty ds\Big[\gamma\frac{\mathrm{e}F\,e^{\mathrm{e}Fs}}{2\sinh(\mathrm{e}Fs)}(x-x')+m\Big]\langle x(s)|x'(s)\rangle &(5.5)\end{aligned}$$

If it is integrable, we'll get the exact solution in constant field!

VI. Plane Wave

In plane wave field, we assume the vector potential of the external field is

$$A_\mu = \varepsilon_\mu f(\xi),$$

where $\xi = n\cdot x,\ n^2=0$ and n is the propagating direction of the wave and $\varepsilon^2=-1$ is the polarization direction. The field tensor is

$$F=\phi f'(\xi),\quad \phi_{\mu\nu}=n_\mu\varepsilon_\nu - n_\nu\varepsilon_\mu,$$

We know $n\cdot\varepsilon=0$ from Maxwell equation so we have $\phi^\mu{}_\nu\phi^\nu{}_\rho = n^\mu n_\rho$ and the equations of motion reduce to

$$\frac{d}{ds}x(s)=2\pi(s)$$
$$\frac{d}{ds}\pi(s)=2\mathrm{e}\phi\pi(s)f'(\xi(s))+\frac{\mathrm{e}}{2}n(\sigma\cdot\phi)f''(\xi(s))$$

We can derive

$$\frac{d}{ds}[n\pi(s)]=0 \quad\Rightarrow\quad n\pi(s)=n\pi(0)=n\pi$$
$$\frac{d}{ds}\xi(s)=2n\pi(s) \quad\Rightarrow\quad \xi(s)-\xi(0)=2n\pi s$$
$$\frac{d}{ds}[\phi\pi(s)]=2\mathrm{e}n(n\pi)f'[\xi(s)]=n\mathrm{e}\frac{d\xi(s)}{ds}f'[\xi(s)]=n\mathrm{e}\frac{d}{ds}f[\xi(s)]$$
$$\Rightarrow\qquad \frac{d}{ds}\{\phi\pi-n\mathrm{e}f[\xi(s)]\}=0,\quad C\equiv\phi\pi-n\mathrm{e}f[\xi(s)]$$
$$\Rightarrow\quad nC=0,\quad \phi C=\phi\phi\pi=n(n\pi),\quad C^2=\pi\phi\phi\pi=(n\pi)^2$$

and the commutators

$$\begin{gathered}[\xi,n\pi]=[nx,n\pi]=in^2=0\\ [\xi(s),\xi(0)]=[\xi(s),\xi(s)-2n\pi s]=0\\ [x(s)-x(0),n\pi]=[x(s),n\pi(s)]-[x(0),n\pi(0)]=0\\ [x(s)-x(0),\xi(s)-\xi(0)]=0\\ [C,n\pi]=[\phi\pi,n\pi]=0\\ [\xi(0),x(s)]=[\xi(s)-2n\pi(s)s,x(s)]=-2ins\end{gathered}\qquad(6.1)$$

We can integrate out π by using C to eliminate the $\phi\pi$ in the equations of motion.

$$\begin{aligned}\frac{d\pi(s)}{ds} &= 2\mathrm{e}\{C+n\mathrm{e}f[\xi(s)]\}f'[\xi(s)]+\frac{d}{d\xi(s)}\{\frac{\mathrm{e}}{2}n(\sigma\cdot\phi)f'[\xi(s)]\}\\ &= \frac{d}{d\xi(s)}\{2\mathrm{e}Cf[\xi(s)]+n\mathrm{e}^2f^2[\xi(s)]+\frac{\mathrm{e}}{2}n(\sigma\cdot\phi)f'[\xi(s)]\},\end{aligned}$$

$$\frac{d}{d\xi(s)}=\frac{1}{2n\pi}\frac{d}{ds}$$

$$\Rightarrow\pi(s)=\frac{1}{2n\pi}\{2\mathrm{e}Cf[\xi(s)]+n\mathrm{e}^2f^2[\xi(s)]+\frac{\mathrm{e}}{2}n(\sigma\cdot\phi)f'[\xi(s)]\}+D \tag{6.2}$$

With π, we can integrate out x from the equations of motion:

$$x(s)-x(0)=\frac{1}{2(n\pi)^2}\int_{\xi(0)}^{\xi(s)}d\zeta[2\mathrm{e}Cf(\zeta)+n\mathrm{e}^2f^2(\zeta)+\frac{\mathrm{e}}{2}n(\sigma\cdot\phi)f'(\zeta)]+2Ds \tag{6.3}$$

(where we convert the s integration into $\zeta=\xi(s)$ integration) Conversely we can also eliminate D in π and represent $n\pi$ in terms of $[\xi(s)-\xi(0)]/s$:

$$\begin{aligned}\pi(s)=&\frac{x(s)-x(0)}{2s}-\frac{s}{[\xi(s)-\xi(0)]^2}\int_{\xi(0)}^{\xi(s)}d\zeta[2\mathrm{e}Cf(\zeta)+\mathrm{e}^2nf^2(\zeta)+\frac{\mathrm{e}}{2}n(\sigma\cdot\phi)f'(\zeta)]\\ &+\frac{s}{\xi(s)-\xi(0)}\{2\mathrm{e}Cf[\xi(s)]+\mathrm{e}^2nf^2[\xi(s)]+\frac{\mathrm{e}}{2}n(\sigma\cdot\phi)f'[\xi(s)]\}\end{aligned} \tag{6.4}$$

At the same time, C can also be expressed as

$$C=\frac{\phi[x(s)-x(0)]}{2s}-\frac{\mathrm{e}n}{\xi(s)-\xi(0)}\int_{\xi(0)}^{\xi(s)}d\zeta f(\zeta) \tag{6.5}$$

Next work is to square π and arrange $x(s)$ ahead of $x(0)$. ($\xi(s)$ and $\xi(0)$ are commute at this time so we can ignor their ordering.) $n^2=0$ and $Cn=0$ make many terms vanish. $x(0)$ can be expressed in terms of $x(s),\pi(s),\xi(s)$ and $\xi(0)$ by (6.4). $x(s)$ commute with $\xi(s)$ and has a eliminated communicator which has n factor with $\xi(0)$ so

$$[x(s),x(0)]=[x(s),-2s\pi(s)]=8is$$

After calculation, H becomes

$$H=\frac{-1}{(2s)^2}[x^2(s)-2x(s)x(0)+x^2(0)]-\frac{2i}{s}+\mathrm{e}^2\langle\delta f^2\rangle+m^2+\frac{\mathrm{e}}{2}(\sigma\cdot\phi)\frac{f[\xi(s)]-f[\xi(0)]}{\xi(s)-\xi(0)}$$

and

$$\langle\delta f^2\rangle=\int_{\xi(0)}^{\xi(s)}d\zeta\frac{f^2(\zeta)}{\xi(s)-\xi(0)}-\Big[\int_{\xi(0)}^{\xi(s)}d\zeta\frac{f(\zeta)}{\xi(s)-\xi(0)}\Big]^2$$

where beside eliminating n^2 and Cn we also substitute and reduce C with (6.5). Using

$$\begin{aligned}[x(s)-x(0)]\phi[x(s)-x(0)] &= [x(s)-x(0)]^\mu(n_\mu\varepsilon^\nu-\varepsilon_\mu n^\nu)[x(s)-x(0)]_\nu\\ &= \varepsilon[\xi(s)-\xi(0),x(s)-x(0)]=0,\end{aligned}$$

$$\int_{\xi(0)}^{\xi(s)} d\zeta f'(\zeta) = f[\xi(s)] - f[\xi(0)]$$

and (3.3), we derived the equations $\langle x(s)|x'(0)\rangle$ satisfies and integrate out

$$\langle x(s)|x'(0)\rangle = \frac{C(x,x')}{s^2}\exp\Big\{-\frac{i(x-x')^2}{4s} - i[\mathrm{e}^2\langle\delta f^2\rangle + m^2 + \frac{\mathrm{e}}{2}(\sigma\cdot\phi)\frac{f(\xi)-f(\xi')}{\xi-\xi'}]s\Big\}$$

where

$$\langle\delta f^2\rangle = \int_{\xi'}^{\xi}\zeta\frac{f^2(\zeta)}{\xi-\xi'} - \Big[\int_{\xi'}^{\xi}\zeta\frac{f(\zeta)}{\xi(s)-\xi(0)}\Big]^2.$$

To fix $C(x,x')$, first we have

$$\begin{aligned}\langle x(s)|\pi(s)|x'(0)\rangle &= \frac{x-x'}{2s} - \frac{s}{(\xi-\xi')^2}\int_{\xi'}^{\xi}d\zeta[2\mathrm{e}Cf(\zeta) + \mathrm{e}^2 nf^2(\zeta) + \frac{\mathrm{e}}{2}n(\sigma\cdot\phi)f'(\zeta)]\\ &+\frac{s}{\xi-\xi'}[2\mathrm{e}Cf(\xi) + \mathrm{e}^2 nf^2(\xi) + \frac{\mathrm{e}}{2}n(\sigma\cdot\phi)f'(\xi)]\end{aligned}$$

$$C = \frac{\phi(x-x')}{2s} - \frac{\mathrm{e}n}{\xi-\xi'}\int_{\xi'}^{\xi}d\zeta f(\zeta)$$

$$\begin{aligned}\partial^x\langle x(s)|x'(0)\rangle &= [\partial^x C(x,x')]\frac{1}{s^2}\exp\{\cdots\} + C(x,x')\frac{1}{s^2}\exp\{\cdots\}(-i)\Big\{\frac{x-x'}{2s}\\ &+s\mathrm{e}^2\Big[\frac{(\xi-\xi')f^2(\xi)n - n\int_{\xi'}^{\xi}d\zeta f^2(\zeta)}{(\xi-\xi')^2} - 2\Big(\int_{\xi'}^{\xi}d\zeta\frac{f(\zeta)}{\xi-\xi'}\Big)\frac{(\xi-\xi')f(\xi)n - n\int_{\xi'}^{\xi}d\zeta f(\zeta)}{(\xi-\xi')^2}\Big]\\ &+s\mathrm{e}\frac{\sigma\cdot\phi}{2}\frac{(\xi-\xi')f'(\xi)n - n[f(\xi)-f(\xi')]}{(\xi-\xi')^2}\Big\}.\end{aligned}$$

Most terms in it will cancel out each other after substituting into the first equation of (4.5). The equation that $C(x,x')$ satisfies is

$$\Big\{i\partial^x - \mathrm{e}A(x) + \frac{\mathrm{e}\phi(x-x')}{(\xi-\xi')}\Big[\int_{\xi'}^{\xi}d\zeta\frac{f(\zeta)}{(\xi-\xi')} - f(\xi)\Big]\Big\}C(x,x') = 0.$$

If we let $s=0$ in (6.2) and eliminate D by (6.3), we get

$$\begin{aligned}\pi(0) &= \frac{x(s)-x(0)}{2s} - \frac{s}{[\xi(s)-\xi(0)]^2}\int_{\xi(0)}^{\xi(s)}d\zeta[2\mathrm{e}Cf(\zeta) + \mathrm{e}^2 nf^2(\zeta) + \frac{\mathrm{e}}{2}n(\sigma\cdot\phi)f'(\zeta)]\\ &+\frac{s}{\xi(s)-\xi(0)}\{2\mathrm{e}Cf[\xi(0)] + \mathrm{e}^2 nf^2[\xi(0)] + \frac{\mathrm{e}}{2}n(\sigma\cdot\phi)f'[\xi(0)]\}\end{aligned}$$

Similarly, we will get another equation by substituting into the second equation of (4.5).

$$\Big\{-i\partial^{x'} - \mathrm{e}A(x') - \frac{\mathrm{e}\phi(x-x')}{(\xi-\xi')}\Big[\int_{\xi'}^{\xi}d\zeta\frac{f(\zeta)}{(\xi-\xi')} - f(\xi')\Big]\Big\}C(x,x') = 0$$

So, we have

$$C(x,x') = \mathrm{C}\exp\Big(-i\mathrm{e}\int_{x'}^{x}dy\Big\{A(y) - \frac{\phi(y-x')}{n\cdot y-\xi'}\Big[\int_{\xi'}^{n\cdot y}d\zeta\frac{f(\zeta)}{n\cdot y-\xi'} - f(n\cdot y)\Big]\Big\}\Big)$$

where C can be fixed by δ-function condition as in constant field case. The integral would reduce to $\int dyA(y)$ if we restricted it on straight line. The final result is

$$\langle x(s)|x'(0)\rangle = \frac{-i}{(4\pi)^2 s^2}\exp\Big\{ -ie\int_{x'}^{x} dy\, A(y) - \frac{i(x-x')^2}{4s} \\ -i[\mathrm{e}^2\langle \delta f^2\rangle + m^2 + \frac{\mathrm{e}}{2}(\sigma\cdot\phi)\frac{f(\xi)-f(\xi')}{\xi-\xi'}]s\Big\}$$

From (4.6), the Dirac propagator is

$$S_F = -i\int_0^\infty ds\Big[\gamma\Big(\frac{x-x'}{2s} - \frac{s}{(\xi-\xi')^2}\int_{\xi'}^{\xi} d\zeta[2\mathrm{e}Cf(\zeta) + e^2 n f^2(\zeta) + \frac{\mathrm{e}}{2}n(\sigma\cdot\phi)f'(\zeta)] \\ + \frac{s}{\xi-\xi'}[2\mathrm{e}Cf(\xi) + \mathrm{e}^2 n f^2(\xi) + \frac{e}{2}n(\sigma\cdot\phi)f'(\xi)]\Big) + m\Big]\langle x(s)|x'(0)\rangle \tag{6.6}$$

VII. Expansion With Respect To e

After above result, we're ready to integrate the last integral to complete the solution. The first ideal is to match with the known result in limiting case. Let $m=0,\ \mathrm{e}=0$:

$$\begin{aligned} S_F &= -i\partial\!\!\!/ i\int_0^\infty ds\frac{-i}{(4\pi)^2 s^2}\exp\frac{-i(x-x')^2}{4s} = -i\partial\!\!\!/\frac{1}{(4\pi)^2}\times\frac{-4}{i(x-x')^2}\exp\frac{-i(x-x')^2}{4s}\Big|_{s=0}^{s=\infty} \\ &= -i\partial\!\!\!/\frac{-i}{(2\pi)^2(x-x')^2} \\ &= \frac{\gamma(x-x')}{2\pi^2(x-x')^4} \end{aligned}$$

Now we turn to constant field. We know the integral can be integrated when $m=0$ and $\mathrm{e}=0$ so we try to expand it with respect to e and see if we can integrate it term by term. The result turns out to be that each order in the expansion of e has the form as $s^{-2}\exp[i(x-x')^2/4s]$ multiplied by polynomials in s:

$$\begin{gathered} U(\mathrm{e}) \equiv \langle x(s)|x'(0)\rangle|_{m=0} \\ U(\mathrm{e}) = \sum_{n=0}^{\infty}\frac{\mathrm{e}^n}{n!}\frac{\partial^n U(\mathrm{e})}{\partial \mathrm{e}^n}\Big|_{\mathrm{e}=0} = U(0) + \mathrm{e}\frac{\partial U}{\partial \mathrm{e}}|_{\mathrm{e}=0} + \frac{\mathrm{e}^2}{2!}\frac{\partial^2 U}{\partial \mathrm{e}^2}|_{\mathrm{e}=0} + \cdots \\ U(0) = \frac{-i}{(4\pi)^2 s^2}\exp\frac{-i(x-x')^2}{4s} \\ \frac{\partial U}{\partial \mathrm{e}} = U(\mathrm{e})\Big\{ -i\int_{x'}^{x} d\xi A(\xi) - \frac{1}{2}\mathrm{tr}(Fs\frac{\partial\ \ln[(\mathrm{e}Fs)^{-1}\sinh(\mathrm{e}Fs)]}{\partial(\mathrm{e}Fs)}) - \frac{i}{2}(\sigma\cdot F)s \\ -\frac{i}{4s}(x-x')Fs\frac{\partial\ \mathrm{e}Fs\coth(\mathrm{e}Fs)}{\partial(\mathrm{e}Fs)}(x-x')\Big\} \end{gathered} \tag{7.1}$$

Or, by expansion formula,

$$\begin{aligned} \ln\frac{\sinh x}{x} &= \tfrac{1}{6}x^2 - \tfrac{1}{180}x^4 + \tfrac{1}{2835}x^6 + \cdots \\ x\coth x &= 1 + \tfrac{1}{3}x^2 - \tfrac{1}{45}x^4 + \tfrac{2}{945}x^6 + \cdots, \end{aligned}$$

we find

$$U'(0) = U(0)\Big[-i\int_{x'}^{x} d\xi A(\xi) - \frac{i}{2}(\sigma\cdot F)s\Big]$$

$$U''(0) = U(0)\Big[-i\int_{x'}^{x} d\xi A(\xi) - \frac{i}{2}(\sigma\cdot F)s\Big]^2 + U(0)[\frac{-s^2}{6}\mathrm{tr}F^2 - \frac{is}{6}(x-x')F^2(x-x')].$$

Arrange in the order of s.

$$U''(0) = U(0)\Big\{-\Big[\int_{x'}^{x} d\xi A(\xi)\Big]^2 - \Big[(\sigma\cdot F)\int_{x'}^{x} d\xi A(\xi) + \frac{i}{6}(x-x')F^2(x-x')\Big]s$$
$$-[\frac{1}{6}\mathrm{tr}F^2 + \frac{1}{4}(\sigma\cdot F)^2]s^2\Big\}$$

We now read the result of first order from (5.4).

$$\begin{aligned} S_F &\approx -(i\partial\!\!\!/ - \mathrm{e}A\!\!\!/)i\int_0^\infty ds[U(0) + eU'(0)] \\ &= -(i\partial\!\!\!/ - \mathrm{e}A\!\!\!/)\frac{1}{(4\pi)^2}\Big\{I_0 - i\mathrm{e}\Big[\int_{x'}^{x} d\xi A(\xi)\Big]I_0 - \frac{i\mathrm{e}}{2}(\sigma\cdot F)I_1\Big\} \end{aligned}$$

where we factor out the integral in s and symbolize it as I_n

$$I_n \equiv \int_0^\infty ds\, s^{n-2}e^{-i\alpha s^{-1}}, \quad \alpha \equiv \frac{(x-x')^2}{4}$$
$$I_0 = \frac{1}{i\alpha} = \frac{4}{i(x-x')^2}$$
$$\frac{\partial}{\partial(-i\alpha)}I_1 = I_0 \quad \Rightarrow \quad I_1 = \ln\alpha + C_1$$

Substitute them into S_F:

$$S_F \approx -(i\partial\!\!\!/ - \mathrm{e}A\!\!\!/)i\Big\{\frac{-i}{(2\pi)^2(x-x')^2}\Big[1 - i\mathrm{e}\int_{x'}^{x} d\xi A(\xi)\Big] - \frac{\mathrm{e}}{2(2\pi)^2}(\sigma\cdot F)[\ln\frac{-(x-x')^2}{4} + C_1]\Big\}$$

Differential it and calculate up to the first order of e:

$$S_F \approx \frac{\gamma(x-x')}{2\pi^2(x-x')^4}\Big[1 - i\mathrm{e}\int_{x'}^{x} d\xi A(\xi)\Big] + \frac{i\mathrm{e}\gamma F(x-x')}{8\pi^2(x-x')^2} + \frac{\gamma(x-x')\mathrm{e}(\sigma\cdot F)}{16\pi^2(x-x')^2}$$

To compare this result with the perturbational one we make $A^\mu = (0, Bx^{(2)}, 0, 0) = g_1^\mu Bx^{(2)}$ in the above equation. This result in

$$F_\mu{}^\nu = \partial_\mu A^\nu - \partial^\nu A_\mu = Bg_\mu^2 g_1^\nu - Bg^{\nu 2}g_{\mu 1}$$
$$\int_{x'}^{x} d\xi A(\xi) = \frac{B}{2}(x-x')_1(x+x')^{(2)}.$$

After reduction:

$$\begin{aligned} S_F \approx\ & \frac{\gamma(x-x')}{2\pi^2(x-x')^4} - \frac{i\mathrm{e}B\gamma(x-x')(x-x')_1(x+x')^{(2)}}{4\pi^2(x-x')^4} \\ & + \frac{i\mathrm{e}B[\gamma^2(x-x')_1 - \gamma_1(x-x')^{(2)}]}{8\pi^2(x-x')^2} + \frac{i\mathrm{e}B\gamma(x-x')[\gamma^2,\gamma_1]}{16\pi^2(x-x')^2} \end{aligned}$$

VIII. Divergent Integral and Infinite Constant

From above discussion, we know that each order of this expansion can be reduced to the linear combination of finite number of I_n's, $(n = 0, 1, \cdots)$. Theoretically, we can integrate out this order by order. But we find that these integrals are all divergent start from I_1. We can get the first order result just because we only use the differential of I_1 and while I_1 is infinite its differential is finite. This give us an ideal to put the infinite part in the integration constants and write the result with these constants.

Define the infinite constants,

$$C_n \equiv \int_0^\infty ds\, s^{n-2} \exp \frac{-i}{s}.$$

We now represent I_n with α and C_n.

$$\frac{\partial}{\partial(-i\alpha)} I_n = I_{n-1} = f(\alpha) \quad \Rightarrow \quad I_n = -i \int^\alpha f(\alpha) = F(\alpha) + \text{Const.}$$

$$I_n\big|_{\alpha=1} = F(1) + \text{Const.} = C_n$$

$$\Rightarrow \quad I_n = F(\alpha) - F(1) + C_n$$

For instance,

$$I_2 = -i \int^\alpha (\ln\alpha + C_1) = -i(\alpha\ln\alpha - \alpha + C_1\alpha) + i(C_1 - 1) + C_2.$$

According to the equality in (7.1), the second order propagator is

$$S_F \approx -(i\partial\!\!\!/ - \mathrm{e}A\!\!\!/)\frac{-i}{(4\pi)^2}\Big\{\Big[1 - i\mathrm{e}\int_{x'}^{x} d\xi A(\xi) - \frac{\mathrm{e}^2}{2}\Big(\int_{x'}^{x} d\xi A(\xi)\Big)^2\Big] I_0$$
$$-\Big[\frac{i\mathrm{e}}{2}(\sigma\cdot F)\Big(1 - i\mathrm{e}\int_{x'}^{x} d\xi A(\xi)\Big) + \frac{i\mathrm{e}^2}{12}(x - x')F^2(x - x')\Big] I_1 - \frac{\mathrm{e}^2}{2}\Big[\frac{1}{6}\mathrm{tr}F^2 + \frac{1}{4}(\sigma\cdot F)^2\Big] I_2\Big\}.$$

The result will be available after substituting $I_0,\ I_1,\ I_2$ and perform the differentiation.

IX. Massive Free Propagator

For further calculations, we should derive the massive Feynman free propagator. Usually, we use contour integral in complex variable and discuss the selection of right pole in (4.2) with residue theory. Since we know that the proper time method has included the appropriate boundary condition, we can directly integrate out the massive propagator from the proper time result.

Let $\mathrm{e} = 0$ in (5.3). We find (5.4) becomes

$$\begin{aligned} S_F &= -(i\partial\!\!\!/ + m)G_F \\ &= -(i\partial\!\!\!/ + m)i\int_0^\infty ds \frac{-i}{(4\pi)^2 s^2} \exp\Big[-\frac{i(x - x')^2}{4s} - im^2 s\Big]. \end{aligned}$$

The scaler propagator part is of the form:

$$I = \int_0^\infty \frac{ds}{s^2} e^{-i\alpha s^{-1} - i\beta s} \tag{9.1}$$

(We let $\alpha = (x - x')^2/4, \quad \beta = m^2$ and multiply it by $1/(4\pi)^2$.)

After the change of variable,

$$\begin{aligned} s = e^z \quad &\Rightarrow \quad ds = e^z dz. \\ s : 0 \to \infty \quad &\Rightarrow \quad z : -\infty \to \infty, \end{aligned}$$

the integral becomes

$$\begin{aligned} I &= \int_{-\infty}^{\infty} dz\, e^{-z} e^{-i\alpha e^{-z} - i\beta e^z} \\ &= i \frac{\partial}{\partial \alpha} \int_{-\infty}^{\infty} dz\, e^{-i\alpha e^{-z} - i\beta e^z} \\ &\equiv i \frac{\partial}{\partial \alpha} I_c. \end{aligned}$$

We notice that the linear combination of e^z and e^{-z} is just the linear combination of $\cosh z$ and $\sinh z$.

$$\begin{aligned} \Rightarrow \quad I_c &= \int_{-\infty}^{\infty} dz\, e^{-i\alpha(\cosh z - \sinh z) - i\beta(\cosh z + \sinh z)} \\ &= \int_{-\infty}^{\infty} dz\, e^{-i[(\beta+\alpha)\cosh z + (\beta - \alpha)\sinh z]} \end{aligned}$$

To combine $\cosh z$ and $\sinh z$ we discuss two cases.

Case I. For $\alpha > 0$, let

$$\begin{aligned} \cosh \theta &\equiv \frac{\beta + \alpha}{\sqrt{(\beta+\alpha)^2 - (\beta - \alpha)^2}} = \frac{\beta + \alpha}{2\sqrt{\beta\alpha}} \\ \sinh \theta &\equiv \frac{\beta - \alpha}{2\sqrt{\beta\alpha}} \end{aligned}$$

then

$$\begin{aligned} I_c &= \int_{-\infty}^{\infty} dz\, e^{-i2\sqrt{\beta\alpha}\cosh(z+\theta)} \\ &= \int_{-\infty}^{\infty} dz\, e^{-i2\sqrt{\beta\alpha}\cosh z} \\ &= -\pi i H_0^{(2)}(2\sqrt{\beta\alpha}) \\ &= -\pi i [J_0(2\sqrt{\beta\alpha}) - i N_0(2\sqrt{\beta\alpha})] \end{aligned}$$

where J, N and H are the Bessel, Neumann and Hankel functions! (see appendix)

Case II. For $\alpha < 0$, let

$$\cosh\theta \equiv \frac{\beta-\alpha}{\sqrt{(\beta-\alpha)^2-(\beta+\alpha)^2}} = \frac{\beta-\alpha}{2\sqrt{-\beta\alpha}}$$
$$\sinh\theta \equiv \frac{\beta+\alpha}{2\sqrt{-\beta\alpha}}.$$

Similarly,

$$\begin{aligned} I_c &= \int_{-\infty}^{\infty} dz\, e^{-i2\sqrt{-\beta\alpha}\sinh(z+\theta)} \\ &= \int_{-\infty}^{\infty} dz\, e^{-i2\sqrt{-\beta\alpha}\sinh z} \\ &= 2K_0(2\sqrt{-\beta\alpha}) \\ &= \pi i[J_0(i2\sqrt{-\beta\alpha}) + iN_0(i2\sqrt{-\beta\alpha})] \\ &= \pi i[J_0(2\sqrt{\beta\alpha}) + iN_0(2\sqrt{\beta\alpha})] \end{aligned}$$

where K is the modified Bessel functions!

Combine these two cases.

$$\begin{aligned} I_c &= -\theta(\alpha)\pi i H_0^{(2)}(2\sqrt{\beta\alpha}) + \theta(-\alpha)2K_0(2\sqrt{-\beta\alpha}) \\ &= -i\pi\epsilon(\alpha)J_0(2\sqrt{\beta\alpha}) - \pi N_0(2\sqrt{\beta\alpha}) \end{aligned}$$

$$\epsilon(x) \equiv \begin{cases} 1, & x > 0 \\ -1, & x \le 0 \end{cases}$$

Therefore,

$$\begin{aligned} I &= i\frac{\partial}{\partial\alpha}I_c = \pi\frac{\partial}{\partial\alpha}[\epsilon(\alpha)J_0(2\sqrt{\beta\alpha}) - iN_0(2\sqrt{\beta\alpha})] \\ &= \pi\delta(\alpha) - \pi\sqrt{\frac{\beta}{\alpha}}[\epsilon(\alpha)J_1(2\sqrt{\beta\alpha}) - iN_1(2\sqrt{\beta\alpha})] \end{aligned}$$

(About the differential respect to x of $N_n(\sqrt{x})$, please refer to the appendix) We can substitute α and β in it.

$$\begin{aligned} G_F &= \frac{1}{(4\pi)^2}\Big\{\pi\delta(\frac{(x-x')^2}{4}) - \frac{2\pi m}{\sqrt{(x-x')^2}}\Big[\epsilon(\frac{(x-x')^2}{4})J_1(m\sqrt{(x-x')^2}) \\ &\quad -iN_1(m\sqrt{(x-x')^2})\Big]\Big\} \\ &= \frac{1}{4\pi}\delta[(x-x')^2] - \frac{m}{8\pi\sqrt{(x-x')^2}}\{\epsilon[(x-x')^2]J_1(m\sqrt{(x-x')^2}) - iN_1(m\sqrt{(x-x')^2})\} \end{aligned}$$

and the Dirac propagator is

$$
\begin{aligned}
S_F &= -i\partial\!\!\!/\Big\{\frac{1}{4\pi}\delta[(x-x')^2] - \frac{m}{8\pi\sqrt{(x-x')^2}}\{\epsilon[(x-x')^2]J_1(m\sqrt{(x-x')^2}) \\
&\quad -iN_1(m\sqrt{(x-x')^2})\}\Big\} - mG_F \\
&= \gamma(x-x')\Big\{\frac{\delta'[(x-x')^2]}{2\pi} - \frac{m^2\delta[(x-x')^2]}{8\pi} + \frac{m^2\epsilon[(x-x')^2]}{8\pi(x-x')^2}J_2(m\sqrt{(x-x')^2}) \\
&\quad -\frac{im^2}{8\pi(x-x')^2}N_2(m\sqrt{(x-x')^2})\Big\} - mG_F
\end{aligned}
$$

The discussions of the δ-functions that appear in above result or whether I_c is defined when α is zero are presented in appendix.

X. The Similarity between Plane Wave Case and Massive Free Propagator

If we investigate the integral in plane wave case we'll find that it is the same with massive propagator except to substitute the β with

$$
\mathrm{e}^2\langle\delta f^2\rangle + m^2 + \frac{\mathrm{e}}{2}(\sigma\cdot\phi)\frac{f(\xi)-f(\xi')}{\xi-\xi'}
$$

instead of m^2. The only labor left is to apply $-(i\partial\!\!\!/ - \mathrm{e}A\!\!\!/ + m)$ on it.

$$
\begin{aligned}
&-(i\partial\!\!\!/ - \mathrm{e}A\!\!\!/ + m)\Big(G_F|_{m^2\to\beta}\Big) \\
&= -i\gamma^\mu(\partial_\mu G_F)|_{m^2=\beta} - i\gamma^\mu\Big(\frac{\partial}{\partial m^2}G_F\Big)\Big|_{m^2=\beta} \times \partial_\mu\beta + (\mathrm{e}A\!\!\!/ - m)\Big(G_F|_{m^2=\beta}\Big)
\end{aligned}
$$

We can use (6.6) and the L_n integral in appendix to derive the result.

$$
\begin{aligned}
S_F(x,x') &= \frac{-1}{(4\pi)^2}\exp\Big\{-ie\int_{x'}^{x}dyA(y)\Big\}\Big(\frac{x-x'}{2}L_{-1} + \frac{1}{\xi-\xi'}\Big\{[2\mathrm{e}Cf(\xi) + \mathrm{e}^2nf^2(\xi) \\
&+\frac{\mathrm{e}}{2}n(\sigma\cdot\phi)f'(\xi)] - \frac{1}{\xi-\xi'}\int_{\xi'}^{\xi}d\zeta[2\mathrm{e}Cf(\zeta) + \mathrm{e}^2nf^2(\zeta) + \frac{\mathrm{e}}{2}n(\sigma\cdot\phi)f'(\zeta)]\Big\}L_1 + mL_0\Big) \quad (10.1)
\end{aligned}
$$

where α and β are

$$
\begin{aligned}
\alpha &= \frac{(x-x')^2}{4} \\
\beta &= \mathrm{e}^2\langle\delta f^2\rangle + m^2 + \frac{\mathrm{e}}{2}(\sigma\cdot\phi)\frac{f(\xi)-f(\xi')}{\xi-\xi'}
\end{aligned}
$$

We skip further reductions.

XI. The Result in Constant Field

We has known the massless propagator in constant field is divergent since second order. After we worked out the integral I in massive propagator, we find the I_0 in just I when β goes to zero. From the ideal of Gaussian Integral, I_n can be derived by differentiating I with respect to β for n times and letting β go to 0. Since we had represented the former result with I_n, we simply replace I_n with L_n (see appendix) in which β is non-zero. Expand (5.5) and use L_n to represent the result after integration.

$$
\begin{aligned}
& \exp\Big[-ie\int_{x'}^{x} d\xi A(\xi)\Big] \\
= \;& 1 - ie\int_{x'}^{x} d\xi A(\xi) - \frac{\mathrm{e}^2}{2}\Big[\int_{x'}^{x} d\xi A(\xi)\Big]^2 + \cdots \\
& \Big[\gamma\frac{\mathrm{e}F e^{\mathrm{e}Fs}}{2\sinh(\mathrm{e}Fs)}(x-x') + m\Big] \\
= \;& m + \frac{\gamma}{2s}(x-x') + \frac{e\gamma}{2}F(x-x') + \frac{\mathrm{e}^2 s\gamma}{6}F^2(x-x') + \cdots \\
& \exp\Big[-\frac{1}{2}\mathrm{tr}\ln\frac{\sinh(\mathrm{e}Fs)}{\mathrm{e}Fs}\Big] \\
= \;& 1 - \frac{\mathrm{e}^2 s^2}{12}\mathrm{tr}F^2 + \cdots \\
& \exp\Big(-\frac{ies}{2}\sigma\cdot F\Big) \\
= \;& 1 - \frac{ies}{2}\sigma\cdot F - \frac{\mathrm{e}^2 s^2}{8}(\sigma\cdot F)^2 + \cdots \\
& \exp\Big[-\frac{i}{4}(x-x')\mathrm{e}F\coth(\mathrm{e}Fs)(x-x')\Big] \\
= \;& \exp\Big[-\frac{i(x-x')^2}{4s}\Big]\Big[1 - \frac{i\mathrm{e}^2 s}{12}(x-x')F^2(x-x') + \cdots\Big]
\end{aligned}
$$

We calculate to the second order of e. Multiplying these expansions and the factor

$$
\frac{-i}{(4\pi)^2 s^2}\exp\Big[-\frac{i}{4s}(x-x')^2 - im^2 s\Big]
$$

S_F is the integration with respect to s. We can write the result order by order.

Zero order:

$$
-\frac{i\gamma}{32\pi^2}(x-x')L_{-1} - \frac{im}{16\pi^2}L_0
$$

First order:

$$
\begin{aligned}
-\frac{\mathrm{e}\gamma}{32\pi^2}(x-x')\Big[\int_{x'}^{x} d\xi A(\xi)\Big]L_{-1} - \frac{i\mathrm{e}}{16\pi^2}\Big[\frac{\gamma}{2}F(x-x') - im\int_{x'}^{x} d\xi A(\xi) \\
-\frac{i\gamma}{4}(x-x')(\sigma\cdot F)\Big]L_0 - \frac{\mathrm{e}m}{32\pi^2}(\sigma\cdot F)L_1
\end{aligned}
$$

Second order:

$$\vdots \tag{11.1}$$

The α and β in L's are.

$$\begin{aligned} \alpha &= \frac{(x-x')^2}{4} \\ \beta &= m^2 \end{aligned}$$

Beside the gauge factor term, we notice the higher order term in the e expansion has higher order of s. That is, we must use higher order of Bessel function to expand the solution.

XII. Conclusion

Generally, we understand that we can approach the solution of the Dirac propagator in external field by perturbation method from free propagator. The integrations are easier only in massless cases and are explicitly integrable only for special external fields.

From the calculations and discussions here, we know that we can use proper time method to calculate the Dirac propagator in external field analytically. We found that the solution of massless propagator are divergent since second order. One the other hand, we found the difference between massive free propagator and the result in plane wave fields is in the substitutions of the mass term. That is, the propagator in plane wave fields is exactly solvable. (See (10.1)).

For constant field, we can't derive a close form solution but we can expand it with respect to coupling constant 'e' and solve it order by order. (See(11.1)) In the calculations, we know every order is convergent and has the form of Bessel functions. It likes to expand the solution by Bessel functions and this guarantees the series is convergent if the solution is smooth.

Appendix

A. Conventions

To reduce the using of notations, we are to explain some conventions here. First, we discuss the contraction of tensor:

1. For rank two tensors $A, B, C\cdots$; and rank one tensors x, y, z

$$\begin{aligned} xy &\equiv x_\mu y^\mu \\ x^2 &\equiv x_\mu x^\mu \\ AB\cdots Cx &\equiv A^\mu{}_{\nu_1} B^{\nu_1}{}_{\nu_2} \cdots C^{\nu_{n-1}}{}_{\nu_n} x^{\nu_n} \\ xAy &\equiv x_\mu A^\mu{}_\nu y^\nu \\ A\cdot B &\equiv A_{\mu\nu} B^{\mu\nu} \end{aligned}$$

2. If the function f can be expanded into polynomials or serials, we can define the matrix function corresponding to the function of rank two tensor:

$$A^n \equiv \overbrace{AA\cdots A}^{n \text{ times}}$$
$$f(s) \equiv \sum_{n=1}^{N \text{ or } \infty} c_n s^n \quad \Rightarrow \quad f(A) \equiv \sum_{n=1}^{N \text{ or } \infty} c_n A^n$$

B. Bessel Functions

The calculations in this report are highly related to Bessel functions. From ordinary text books, we can find the following formulas for Bessel functions. (Where Ω_n can be Bessel, Neumann, Modified Bessel or Hankel functions.)

Generating Equation

$$J_n(x) = (-1)^n x^n \left(\frac{d}{x\,dx}\right)^n J_0(x)$$

Recursive Formula

$$\Omega_{n-1}(x) + \Omega_{n+1}(x) = \frac{2n}{x}\Omega_n(x)$$
$$\Omega_{n-1}(x) - \Omega_{n+1}(x) = 2\Omega_n'(x)$$

Integral Transforming Formula

$$H_0(x) = \frac{i}{\pi}\int_{-\infty}^{\infty} dz e^{-ix\cosh z}$$
$$K_0(x) = \frac{1}{2}\int_{-\infty}^{\infty} dz e^{-ix\sinh z}$$

Expansions

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(s+n)!}\left(\frac{x}{2}\right)^{n+2s}$$

C. The Differential of N_n

Because $N_n(\sqrt{x})$ is complex function, we have to make more careful discussions of it's differential with respect to x at origin.

We know that $\ln x$ is a function defined on positive real number with $1/x$ as it's differential. While we continuously extend it over the whole complex plane, it is discontinue at the origin. The amount of discontinuity while taking principle value is

$$\ln\epsilon - \ln(-\epsilon) = -\ln(-1) = -i\pi$$

but the integral of $1/x$ doesn't reflect this. So, we have to modify the differential of $\ln x$ into

$$\frac{d}{dx}\ln x = \frac{1}{x} - i\pi\delta(x).$$

The $N_n(\sqrt{x})$ expansion becomes (refer to "Mathematical Methods for Physics" by Arfken)

$$\begin{aligned} N_n(\sqrt{x}) &= \frac{1}{\pi}J_n(\sqrt{x})\ln\frac{x}{4} - \frac{1}{\pi}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!(n+r)!}(\frac{x}{4})^{\frac{n}{2}+r}[\mathcal{F}(r)+\mathcal{F}(n+r)] \\ &\quad -\frac{1}{\pi}\sum_{r=0}^{n-1}\frac{(n-r-1)!}{r!}(\frac{x}{4})^{-\frac{n}{2}+r} \end{aligned}$$

Only the principle value of ln has discontinuity at origin so the modification term we have to add is $-iJ_n(\sqrt{x})\delta(x)$

That is,

$$\frac{d}{dx}N_n(\sqrt{x}) = \frac{\sqrt{x}}{4}[N_{n-1}(\sqrt{x}) - N_{n+1}(\sqrt{x})] - iJ_n(\sqrt{x})\delta(x).$$

This formula was used above while taking the integral of I and solving Dirac propagator by differentiation to derive correct factor of δ-function. (We surely can cover the δ-function by redefining N_n but this results in a different form of solution from most text books.)

**If we define

$$S_n(\sqrt{x}) \equiv \theta(-x)J_n(\sqrt{x}) + iN_n(\sqrt{x}),$$

the δ-function resulting from the differentiation of the θ-function will just cancel that of N_n.

$$\frac{d}{dx}S_n(\sqrt{x}) = \frac{1}{4\sqrt{x}}[S_{n-1}(\sqrt{x}) - S_{n+1}(\sqrt{x})]$$

It has no δ-function here!

D. L_n Integral

If we didn't let $m = 0$ in above expansion for constant field, we would have the integral,

$$\begin{aligned} L_n &\equiv \int_0^{\infty} ds\, s^{n-2}e^{-i\alpha s^{-1} - i\beta s} \\ &= i^n\frac{\partial^n}{\partial\beta^n}I \\ &= i^n\frac{\partial^n}{\partial\beta^n}\Big\{\pi\delta(\alpha) - \pi\sqrt{\frac{\beta}{\alpha}}[\epsilon(\alpha)J_1(2\sqrt{\beta\alpha}) - iN_1(2\sqrt{\beta\alpha})]\Big\} \end{aligned}$$

Let's assume $\alpha \neq 0, \quad \beta > 0$ and calculate some L_n's

$$
\begin{aligned}
L_0 &= -\pi\sqrt{\frac{\beta}{\alpha}}[\epsilon(\alpha)J_1(2\sqrt{\beta\alpha}) - iN_1(2\sqrt{\beta\alpha})] \\
L_1 &= \frac{-i\pi}{2\sqrt{\beta\alpha}}[\epsilon(\alpha)J_1(2\sqrt{\beta\alpha}) - iN_1(2\sqrt{\beta\alpha})] - i\pi[\epsilon(\alpha)J_1'(2\sqrt{\beta\alpha}) - iN_1'(2\sqrt{\beta\alpha})] \\
&= -i\pi[\epsilon(\alpha)J_0(2\sqrt{\beta\alpha}) - iN_0(2\sqrt{\beta\alpha})] \\
L_2 &= -\pi\sqrt{\frac{\alpha}{\beta}}[\epsilon(\alpha)J_1(2\sqrt{\beta\alpha}) - iN_1(2\sqrt{\beta\alpha})] \\
&\vdots
\end{aligned}
$$

After observing the leading L_n's, we can proof the general form:

$$
L_n = (-i)^n\pi\left(\sqrt{\frac{\alpha}{\beta}}\right)^{n-1}[\epsilon(\alpha)J_{n-1}(2\sqrt{\beta\alpha}) - iN_{n-1}(2\sqrt{\beta\alpha})] \tag{D.1}
$$

We turn to the explicit form of L_n where n is any integer. We started from L_0, that is I, in the above calculations but the most fundamental (or simplest) one is L_1, that is I_c. Therefore, we can generate L_n for $n > 1$ by differentiating L_1 with respect to β or for $n < 1$ with respect to α.

$$
\begin{aligned}
L_1 &\equiv \int_0^\infty ds\, s^{-1}e^{-i\alpha s^{-1}-i\beta s} \\
&= -i\pi[\epsilon(\beta\alpha)J_0(2\sqrt{\beta\alpha}) - iN_0(2\sqrt{\beta\alpha})] \\
&= -i\pi[\theta(\beta\alpha)J_0(2\sqrt{\beta\alpha}) - S_0(2\sqrt{\beta\alpha})]
\end{aligned}
$$

(The benefit of using S is the convenience in the calculation of δ-functions term) Here we don't assume positive β so we have to add the β factor in the ϵ-function or θ-function. (This can be observed from the symmetry of α and β under the transformation, $s \to s^{-1}$.)

$$
L_n = \begin{cases} i^{n-1}\left(\frac{\partial}{\partial\beta}\right)^{n-1}L_1, & \text{for} \quad n \geq 1 \\ i^{1-n}\left(\frac{\partial}{\partial\alpha}\right)^{1-n}L_1, & \text{for} \quad n < 1 \end{cases}
$$

$$
\Rightarrow \qquad L_{-n} = L_{n+2}\Big|_{\beta\leftrightarrow\alpha}
$$

We only need to see the cases for $n > 1$:

$$
\begin{aligned}
L_n &= i^{n-1}\left(\frac{\partial}{\partial\beta}\right)^{n-1}L_1 \\
&= (-i)^{n-1}\left(\sqrt{\frac{\alpha}{\beta}}\right)^{n-1}x^{n-1}\left(\frac{\partial}{x\,\partial x}\right)^{n-1}L_1, \qquad \text{mist.} \quad x \equiv 2\sqrt{\beta\alpha} \\
&= \pi(-1)^n\left(\sqrt{\frac{\alpha}{\beta}}\right)^{n-1}[\theta(\beta\alpha)J_{n-1}(2\sqrt{\beta\alpha}) - S_{n-1}(2\sqrt{\beta\alpha})] - i\pi(\delta\text{-function term})
\end{aligned}
$$

where the δ-function term, after calculations, is

$$(\delta\text{-function term}) = (i\alpha)^{n-1} \sum_{k=0}^{n-2} \delta^{n-2-k}(\beta\alpha) \frac{(n-1)!}{k!(n-2-k)!} \sum_{j=0}^{k} \frac{(-1)^j}{j!(k-j)!} \frac{1}{n-1-j}$$

With these result, the former expansions for arbitrary orders are merely algebraic calculations without any integral or differentiation!

E. References

George Arfken, *"Mathematical Methods for Physics,"* Third Edition. Academic, New York, (1985).

Nikolai Nikolaevich Bogoliubov & Dmitrii V. Shirkov, *"Introduction to the Theory of Quantized Fields."* Interscience Publishers, New York, (1959).

Paul Adrian Maurice Dirac, *"The Principles of Quantum Mechanics,"* Fourth Edition. Oxford University Press, London, (1958).

Claude Itzykson & Jean-Bernard Zuber, *"Quantum Field Theory."* McGraw-Hill Book Company, Singapore (1985).

Julian Schwinger, *"On Gauge Invariance and Vacuum Polarization."* Phys. Rev. **82**, 664 (1951).

Henry William Wyld, *"Mathematical Method for Physics."* Benjamin, Reading, Massachusetts, (1976).